How fast does a continued fraction converge?

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Abstract

Did you know that when the continued fraction $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots$ converges, the infinite series $\sum_{n=1}^{\infty}|a_n|$ diverges? And that the divergence of the series is necessary but not sufficient! In this paper we discuss some of the interesting properties from the classical studies of continued fractions, cfs. We study complex and periodic cfs. Of special interest is the cf in which the period is 1. Let b be any positive number and $u(b)$ be the positive root of the quadratic equation $x^2 - bx - 1 = 0$. We show how to compute the rate at which the cf with all $a_n = b$ converges to $u(b)$. It turns out that this rate depends only upon b and not upon whether the limit is rational or irrational. When the partial numerators settle down to only $1s$, we get infinitely many so-called *most irrational* numbers.

1 Introduction

In mathematics we often encounter unending processes such as infinite sequences, infinite series, infinite products, continued fractions, continued radicals, and so forth. We may ask questions such as: Under what conditions do these processes produce a finite answer? How many steps will be necessary to achieve an approximation to within a given error? In other words, how fast does this process converge? And in computer sciences we sometimes ask: Is there another process that will produce as good or an even better answer in fewer steps?

In this paper one of the questions we want to answer is: If b is a positive number, and $b \neq 1$, how slowly does the continued fraction

$$x = b + \cfrac{1}{b + \cfrac{1}{b + \ddots}} \tag{1}$$

converge compared to how slowly

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}} \tag{2}$$

converges?

We will find that when $b > 1$, (1) converges more rapidly than (2), and when $0 < b < 1$, (1) is slower than (2). This latter result is logical enough, but it is somewhat surprising in the face of claims that (2) is the "slowest converging" continued fraction. Livio[3, pp 114]. The conflict is resolved when we realize that the aforementioned claim is based upon considering only special cases in which the values of b are restricted to the whole numbers. Historically, the domains of continued fractions include all real and complex numbers and are not restricted to just the positive integers.

If $a_0, a_1, a_2, ...$ and $b_1, b_2, b_3, ...$ are any complex numbers, we write the continued fraction

$$a_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{3}$$

in the more compact and easier to read form

$$a_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} .... \tag{4}$$

The convergence or divergence of (4) depends upon whether or not the the $a_p s$ and the $b_p s$ meet certain conditions. In addition, when the $a_p s$ are non-zero, we can further simplify the notation, without loss of generality, by rewriting (4) as follows.

$$c_0 + \frac{1}{c_1 +} \frac{1}{c_2 +} \frac{1}{c_3 +} ... \tag{5}$$

where $c_0 = a_0$, $c_1 = b_1/a_1$, $c_2 = (a_1 b_2)/a_2$, $c_3 = (a_2 b_3)/(a_1 a_3)$, etc. When the $c_p s$ are real, and are all of the same sign, (5) converges if and only if the infinite series $\sum |c_p|$ diverges.

In (5), if the $c_p s$ are complex we need the following additional requirement. Let $\mathcal{R}(c_p)$ and $\mathcal{I}(c_p)$ be the real and imaginary parts of c_p, respectively, and suppose there exists a positive number, k and a positive integer N such that $|\mathcal{I}(c_p)| < k\,|\mathcal{R}(c_p)|$ for all $p > N$, then (5) converges. Van Vleck[4]. This implies, for example, that if $\mathcal{R}(c_p) = 0$ for all p, then (5) does not converge even if $\sum |c_p|$ diverges.

Example 1 *If z is the complex number, $z = i$, then the continued fraction*

$$z + \frac{1}{z+}\frac{1}{z+}\cdots$$

does not converge.

Example 2 *If we let P and Q be real numbers with $P \neq 0$, and $z = P + iQ$, then the continued fraction*

$$z + \frac{1}{z+}\frac{1}{z+}\cdots$$

does converge and its limit is

$$\frac{1}{2}(P + A) + i\frac{1}{2}(Q + B), \tag{6}$$

where

$$\begin{aligned} A &= \sqrt{\frac{\sqrt{K^2 + L^2} + K}{2}}, \\ B &= \sqrt{\frac{\sqrt{K^2 + L^2} - K}{2}}, \\ K &= \mathcal{R}(z^2) + 4, \\ L &= \mathcal{I}(z^2). \end{aligned}$$

This follows from Van Vleck.

2 Linear Fractional Transformations

Let a_p and b_p be any complex numbers. Wall[5,pp13], defines "...an infinite sequence of linear fractional transformations, $\tau_p(w)$ of the variable w into the variable τ" as follows:

$$\begin{aligned} \tau_0(w) &= b_0 + w \\ \tau_p(w) &= \frac{a_p}{b_p + w}. \end{aligned} \tag{7}$$

If we denote the composite of these functions $\tau_i[\tau_j...[\tau_k(w)]]$ by $\tau_i\tau_j...\tau_k(w)$, we will get the identity

$$\tau_0\tau_1...\tau_n(0) = \tau_0\tau_1...\tau_{n+1}(\infty).$$

The linear fractional transformation (7) can be written as

$$\tau_0\tau_1...\tau_n(w) = \frac{A_{n-1}w + A_n}{B_{n-1}w + B_n} \tag{8}$$

where A_n and B_n are functions of a_p and b_p. This can be proved by induction starting with $A_{-1} = 1, B_{-1} = 0, A_0 = a_0, B_0 = 1$, and using the following fundamental **recurrence formulas.**

$$\begin{pmatrix} A_{p+1} \\ B_{p+1} \end{pmatrix} = \begin{pmatrix} A_p & A_{p-1} \\ B_p & B_{p-1} \end{pmatrix} \begin{pmatrix} b_{p+1} \\ a_{p+1} \end{pmatrix}, \text{ for } p = 0, 1, 2, 3, ...$$

We compute a few terms getting

$$\begin{aligned} \frac{A_0}{B_0} &= a_0,\ \frac{A_1}{B_1} = a_0 + \frac{a_1}{b_1},\ \frac{A_2}{B_2} = a_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2}}, ... \\ \lim_{n\to\infty} \frac{A_n}{B_n} &= a_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots \end{aligned} \tag{9}$$

which corresponds to (4).

When the b_ps, the denominators, are non-zero we can convert this continued fraction into the following form

$$d_0 + \frac{d_1}{1 +} \frac{d_2}{1 +} \frac{d_3}{1 +} .... \tag{10}$$

In 1865 Worpitzky[6] proved that (10) converges if there exists a positive integer P such that $|d_p| \le \frac{1}{4}$, for all $p > P$. Later, in 1939, Leighton[2] proved that (10) converges if the ratio $|d_{p+1}/d_p| \le k < 1$ for all sufficiently large p. The cf (10) could converge or diverge if the limit of the ratio is 1.

3 Periodic Continued Fractions

If the partial denominators $c_1, c_2, \ldots$ in (5) exhibit the following repeating pattern

$$a + \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots \frac{1}{b_n +} \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots$$

then this cf is said to be a periodic continued fraction of period n.

When the $c_p s$ are real and have the same sign, the periodic continued fraction, pcf always converges because

$$\sum_{p=1}^{\infty} |b_p|$$

diverges.

But when the $c_p s$ have non zero imaginary parts, the Van Vleck criteria are needed for convergence.

3.1 Formula for pcf of period 1

If the repeating pattern in a periodic cf is of length 1, then its limit is just the positive root of the quadratic equation $x^2 - bx - 1 = 0$

$$\begin{aligned} & b + \frac{1}{b +} \frac{1}{b +} \frac{1}{b +} \cdots \\ = \quad & \frac{b + \sqrt{b^2 + 4}}{2}. \end{aligned}$$

3.2 Formula for pcf of period 2

If a is any real number and the period $n = 2$, and the repeating terms are any positive real numbers, p and q (not necessarily distinct), then the limit of

$$a + \frac{1}{p +} \frac{1}{q +} \frac{1}{p +} \frac{1}{q +} \cdots$$

is

$$= \frac{1}{2}(2a - q) + \frac{1}{2}\sqrt{q^2 + 4q/p}.$$

These formulas also hold when $b, p,$or q are complex numbers that satisfy the Van Vleck condition.

3.3 Square roots as $pcfs$ with integral partial denominators

It is interesting to note that if N is a positive integer and not a perfect square then $\sqrt{N}$ can be written as a pcf in which the $c_p s$ are integers.

Example:

$$\sqrt{11} = 3 + \frac{1}{3+}\frac{1}{6+}\frac{1}{3+}\frac{1}{6+}\cdots.$$

4 How to measure how fast the pcf of period 1 converges

4.1 Wall's Formula

Let b be any positive real number. Denote by $u(b)$ and $v(b)$ respectively, the positive and negative roots of the quadratic equation

$$x^2 - bx - 1 = 0. \tag{11}$$

Following Wall[5, pp. 23], for all positive integers n, we define $w_n(b)$, the nth approximant of the continued fraction, cf, converging to $u(b)$ as follows

$$w_n(b) = u(b) + v(b) - \frac{v(b)}{\frac{v(b)}{u(b)} + 1/\sum_{p=0}^{n-1}(v(b)/u(b))^p}. \tag{12}$$

A generalizatiom of this and other related formulas are due to Euler[1], but we will consider only this simple version. Let's call Equation (12) "Wall's Formula." We prove that it converges and we determine its rate of convergence for every real number $b > 0$. In addition, we will show that:

- The rate of convergence depends only upon b,
- This rate increases as b increases, and
- The rate is exactly the same as the rate of convergence of (1).

$$b + \frac{1}{b+}\frac{1}{b+}\frac{1}{b+}\cdots$$

First, let's prove that Wall's Formula converges to $u(b)$ and compute its rate of convergence. Since $u(b) = (b+\sqrt{b^2+4})/2$ and $v(b) = (b-\sqrt{b^2+4})/2$ are the roots to the quadratic equation (10), then it is easy to prove that

$$\left|\frac{v(b)}{u(b)}\right| = \left(\frac{1}{u(b)}\right)^2 < 1. \tag{13}$$

The inequality in (13) tells us that the geometric series

$$\sum_{p=0}^{n-1} (v(b)/u(b))^p = \frac{1-\left(\frac{v(b)}{u(b)}\right)^n}{1-\frac{v(b)}{u(b)}} \tag{14}$$

is absolutely convergent, and its limit is

$$\frac{u(b)}{u(b)-v(b)}.$$

Therefore

$$\begin{aligned}\lim_{n\to\infty} w_n(b) &= u(b) + v(b) - \frac{v(b)}{\frac{v(b)}{u(b)} + \frac{u(b)-v(b)}{u(b)}} \\ &= b - v(b) = u(b).\end{aligned}$$

4.2 Why convergence is faster for larger b

When b increases what happens to the ratio $\left|\frac{v(b)}{u(b)}\right|$?

Solution:

1. If $b_1 > b_2$ and both are positive, then $u(b_1) > u(b_2)$,

$$\begin{aligned}\frac{1}{u^2(b_1)} &< \frac{1}{u^2(b_2)}, \text{ or} \\ \left|\frac{v(b_1)}{u(b_1)}\right| &< \left|\frac{v(b_2)}{u(b_2)}\right|.\end{aligned}$$

That is, the ratio decreases as b increases.

The larger the value of b, the smaller the ratio, thus the faster the geometric series converges. Since this series is in the denominator of the denominator of Wall's Formula, then $w_n(b)$ converges to $u(b)$ faster for larger b.

This result is independent of whether or not $u(b)$ is rational. Thus, for example

$$\begin{aligned} w_n(1) \text{ converges to } u(1) &= \frac{1+\sqrt{5}}{2}, \text{ faster than} \\ w_n(\frac{7}{12}) \text{ converges to } u(\frac{7}{12}) &= \frac{4}{3}. \end{aligned}$$

Let us write the continued fraction

$$b + \frac{1}{b+}\frac{1}{b+}\frac{1}{b+}\cdots$$

recursively in order to compare this with Wall's Formula.

$$\begin{aligned} c_0 &= b \\ c_n(b) &= b + 1/c_{n-1}(b), \text{ for } n \geq 1. \end{aligned} \tag{15}$$

4.3 Relation between $w_n(b)$ and $c_n(b)$

We now state a remarkable equation, for $n \geq 1$.

$$w_n(b) = \frac{u^{n+2}(b) - v^{n+2}(b)}{u^{n+1}(b) - v^{n+1}(b)}. \tag{16}$$

An interesting proof of this can be obtained by starting with (12), substituting in (14) and making use of the fact that $u(b)$ and $v(b)$ are roots of the quadratic equation (11). Now by induction, we can prove that Wall's Formula, (12) is the same as the pcf in equation (15). That is

$$c_n(b) = w_n(b). \tag{17}$$

Equation (17) tells us that the rate at which $c_n(b)$ converges increases as b increases. Therefore, if $b < 1$, then

$$b + \frac{1}{b+}\frac{1}{b+}\frac{1}{b+}\cdots$$

converges to $(b+\sqrt{b^2+4})/2$ more slowly than

$$1+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$$

converges to ϕ.

4.4 All irrationals are in the range of u(b) and v(b)

Both Wall's formula $w_n(b)$ and the period-1 pcf, $c_n(b)$ have the same limit, namely

$$\lim_{n\to\infty}\frac{A_n}{B_n}=\frac{(b+\sqrt{b^2+4})}{2}$$

which is the positive root of the quadratic equation $x^2-bx-1=0$. Now, since we are allowing b to be any positive real number, every positive real number is in the range of $u(b)$. And every negative real is in the range of $v(b)$. See Figure 1.

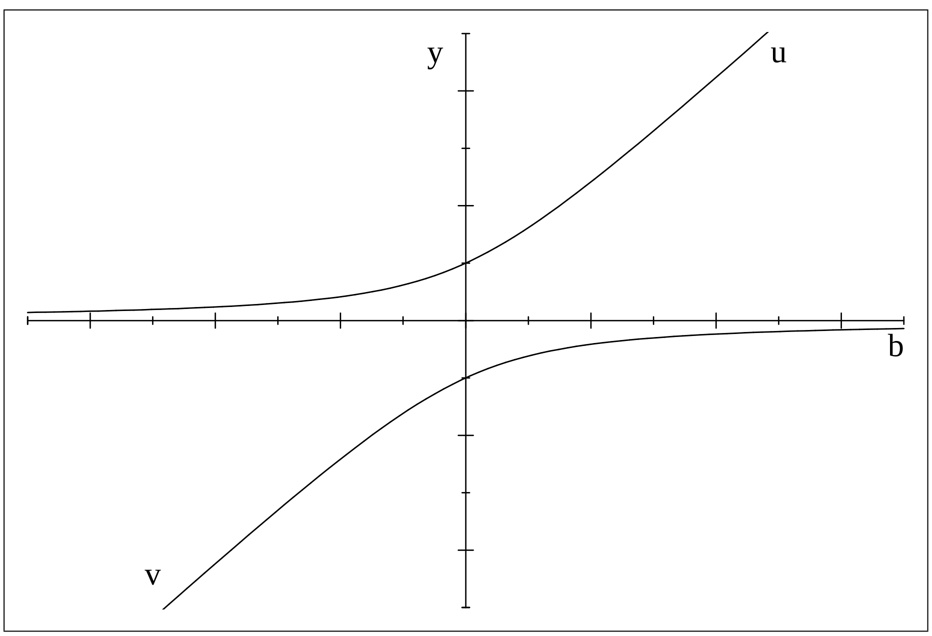

Figure 1 graphs of $u(b)$ and $v(b)$. Range is all reals $\neq 0$

To solve the inverse problem, we start with any (irrational or rational) number, $y\neq 0$, then find what b it takes to get y as a root. Just define b as $y-1/y$. The graph is shown in Figure 2.

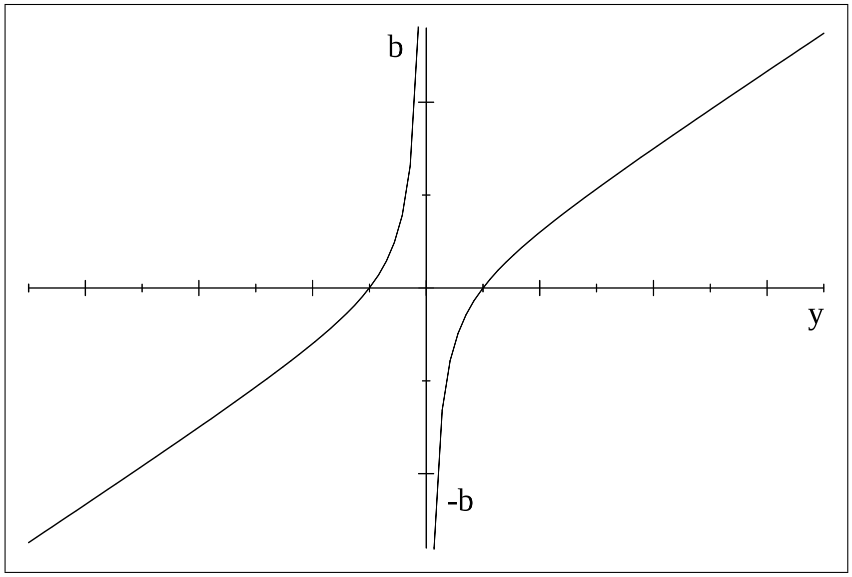

Figure 2 Graph of $b(y)$. Horizontal axis is y

These graphs are inverses of each other. Thus, $y = u(b)$, or $-1/u(b)$, and $b = u(b) - 1/u(b)$. See the combined graphs in Figure 3

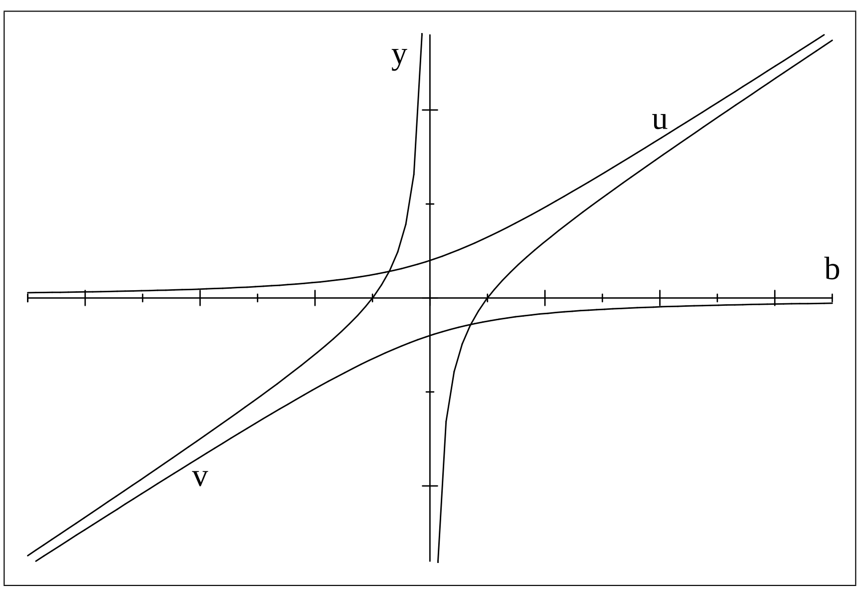

Figure 3. Showing how b is related to $u(b)$ and $v(b)$

5 Continued Fractions with integral partial denominators.

An *integral continued fraction, icf*, is defined as the cf (5), in which all of the partial denominators, $c_p s$ are positive (or negative) integers. Then

1. The cf will converge to a limit x, and
2. Any truncation will yield a rational approximation of x.

Thus, if

$$x = c_0 + \frac{1}{c_1 +} \frac{1}{c_2 +} \frac{1}{c_3 +} \cdots$$

and, for some integer n,

$$x_n = c_0 + \frac{1}{c_1 +} \frac{1}{c_2 +} \cdots \frac{1}{+ c_n},$$

then x_n is a rational approximation of x. We know that the icf converges because all of the $c_p s$ are positive or negative integers, so $\sum |c_p|$ diverges. Every irrational number x can be written as such an icf.

Example 3 *Consider the following icf*

$$\pi^2 = 9 + \frac{1}{1 +} \frac{1}{6 +} \frac{1}{1 +} \frac{1}{2 +} \frac{1}{47 +} \cdots \tag{18}$$

Cutting this off after just one term we get $x_0 = 9$. *After two terms,* $x_1 = 10$, *then* $x_2 = 69/7$, $x_3 = 79/8$, $x_4 = 227/23$, *etc.*

In the next section we will state an algorithm for getting $icfs$ for irrational numbers.

6 Nearest Rational Approximation

The concept of "nearest rational approximation" is used as a measure of how irrational a number is.

6.1 What is the nearest rational approximation of an irrational number?

We take the following as our definition.

Definition 4 *If $x > 0$ is any irrational number and $\epsilon > 0$ is any prescribed error, then the rational fraction $\frac{A}{B}$ with the smallest denominator such that $\left|x - \frac{A}{B}\right| < \epsilon$ is the nearest rational approximation, nra, to x for this ϵ.*

If we compare two irrational numbers x and y and we want to say "x is more irrational than y" then, according to this definition, we need to show that the rational numbers approximating x to a within a certain error, must have larger denominators than those rationals approximating y to within the same error.

A frequently used example is the comparison of π to ϕ. Say we want an approximation with error $\epsilon = 0.002$. We note that $\left|\pi - \frac{22}{7}\right| < \epsilon$, but in order to find a rational number that comes within this same ϵ for ϕ, we need the fraction $\frac{34}{21}$. That is, $\left|\phi - \frac{34}{21}\right| < \epsilon$, and for no other rational fraction with a denominator less than 21 will this be true.

6.2 Finding the nra

The algorithm for finding the nra is to construct a continued fraction by manipulating a decimal expression as follows.

Let N be any real positive number written in decimal form, rational or irrational, thus:

$$N = b_0.a_1a_2a_3...,$$

where b_0 is a nonnegative integer and each a_p is a base-ten digit.

Write N as the sum of its integral part and its fractional part as follows:

$$\begin{aligned} N &= b_0 + 0.a_1a_2a_3... \\ &= b_0 + \frac{1}{1/(0.a_1a_2a_3...)}. \end{aligned}$$

Note that if $0.a_1a_2a_3...$is zero then N is just the integer b_0, and we terminate the process with $N = b_0$. Carry out the division $1/(0.a_1a_2a_3...)$ This produces a decimal of the form $b_1.c_1c_2c_3...$, where b_1 is an integer, $b_1 \geq 1$, and

c_p is a base-ten digit. We know that $b_1 \geq 1$ because $0.a_1a_2a_3... < 1$. Now if $0.c_1c_2c_3...$ is not zero, we can write

$$N = b_0 + \frac{1}{b_1 + 1/(1/(0.c_1c_2c_3...))}, \text{ etc.}$$

If this process terminates, the fractional part becomes zero, and N is a rational number. Otherwise the process continues. If you truncate the fraction at some stage, you will have a continued fraction in which all the $b_p s$ are positive integers and when simplified will yield a rational number $\frac{A(n)}{B(n)}$ approximation of N. No rational number whose denominator is less than $B(n)$ will be as good an approximation to N.

The expression $1/\ 0.c_1c_2c_3...$ can never be less than 1, all the $b_p \geq 1$ and none of the *nracf* can converge more slowly than the one in which every $b_p = 1$. Thus,

$$N = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$$

is one of the slowest converging *nracf*. This N is actually ϕ. Because all of the denominators are 1 in this *nracf*, then ϕ could be called one of the most irrational numbers between 1 and 2.

Likewise, the number N_2 defined as

$$N_2 = 2 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$$

is one of the most irrational numbers between 2 and 3 etc. It is clear that the *nracf* for N_2 converges just as slowly as the one for ϕ.

If N is any positive integer then the irrational number

$$y = N + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$$

is also one of the slowest converging *nracf* This is because all of the denominators of y are $1s$, therefore y converges just as slowly as ϕ Actually $y = N + \frac{1}{\phi}$.

Thus, there are infinitely many counter-examples to the statement that ϕ is the most irrational number; just take any positive integer M and add $\frac{1}{\phi}$. In addition you can create more such counter examples by adding integers to the reciprocals of these, for example

$$\lambda_1 = 1 + \frac{1}{M + \frac{1}{\phi}}.$$

This converges just as slowly as ϕ, but then so does

$$\begin{aligned}\lambda_2 &= 1+\frac{1}{P+\frac{1}{\phi}} \text{ for any other positive integer, } P, \\ \lambda_3 &= 1+\frac{1}{\lambda_1}, \text{ and} \\ \lambda_4 &= 1+\frac{1}{\lambda_2} \text{ or} \\ \lambda_5 &= 1+\frac{1}{\lambda_3}, \text{ etc.}\end{aligned}$$

Any such combination yields $\lambda's$ that are between 1 and 2, and there are also infinitely many equally irrational numbers in every interval. For example in the interval $[2,3]$,the number 2.232791689... is just as irrational as ϕ, because it is defined as follows.

$$2.232791689... = 2+\frac{1}{4+}\frac{1}{3+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\cdots$$

7 References

1. Euler, L. *Introductio in analysin infinitorium*, Chapter 18, Vol. 1, 1748

2. Leighton, Walter "A test-ratio for continued fractions," *Bull. Amer. Math. Soc.* vol 45(1939), pp 97-100.

3. Livio, M. *The Golden Ratio: The Story of Phi, the World's Most Astonishing Number*, Broadway Books, 2002.

4. Van Vleck, E. V. "On the convergence of continued fractions with complex elements," *Trans. Amer. Math. Soc.*, vol 2(1901), pp. 215-233.

5. Wall, H. S. *Analytic Theory of Continued Fractions,* Van Nostrand, Co. 1948, Reprinted by AMS Chelsea Publishing, 2000.

6. Worpitzky, J. "Untersuchungen uber die Entwickelung der monodronen und monogenen Funktionen durch Kettenbruche," Friedrichs-Gymnasium *und Realschule*, Jahresbericht, Berlin, 1865, pp. 3-29.