Not a Proof

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1 Excerpt from *Generalizations of the Golden Ratio*

Why is the continued fraction for phi not a proof that *phi is the most irrational number?* For any positive real number y, we define the generalized golden ratio ϕ_y as the positive solution to the quadratic equation

$$(\phi_y)^2 - y\phi_y - 1 \quad = \quad 0, \text{ that is,} \tag{1}$$

$$\phi_y \quad = \quad \frac{y + \sqrt{y^2 + 4}}{2}. \tag{2}$$

Note that $\phi_1 = \phi$.

From Equation (1)

$$\phi_y \quad = \quad y + \frac{1}{\phi_y}, \text{ and} \tag{2a}$$

$$\phi_y \quad = \quad \sqrt{1 + y\phi_y}. \tag{2b}$$

Equations (2a) and (2b) can be used to prove, for any positive number y, the following continued fraction and the nested roots equations

$$\phi_y \quad = \quad y + \cfrac{1}{y + \cfrac{1}{y + \frac{1}{y+1}\dots}}, \tag{3}$$

$$\phi_y \quad = \quad \sqrt{1 + y\sqrt{1 + y\sqrt{1 + y\sqrt{1 + \dots}}}} \tag{4}$$

When $y > 1$, the continued fraction in Equation(3), converges more rapidly than it would were $y = 1$. For this reason, the continued fraction for $y = 1$ is said to produce an irrational number that is the "most irrational number." See Livio[1]. Unfortunately, if $0 < y < 1$ the continued fraction for ϕ_y converges more slowly than for ϕ_1. We can illustrate the various rates of convergence by graphs. (I owe thanks to my wife Jean for giving me this idea.) To be precise, let us define a sequence of continued fractions for a given y as follows

$$cf_1(y) \quad = \quad y$$

$$cf_n(y) \quad = \quad y + \frac{1}{cf_{n-1}(y)},$$

where n is a positive integer, and $n > 1$.

Thus, when $y = 1$, $\{cf_n(1)\}_{n=1}^{\infty}$ is the continued fraction for ϕ,

$$\lim_{n \to \infty} cf_n(1) = \phi.$$

Does $cf_n(1)$ converge more slowly than $cf_n(2)$? Yes, as can be seen in Figure 1, the dashed line graph shows that it takes more terms of $cf_n(1)$ to settle down onto ϕ_1, than it does for the graph of $cf_n(2)$ to settle onto ϕ_2.
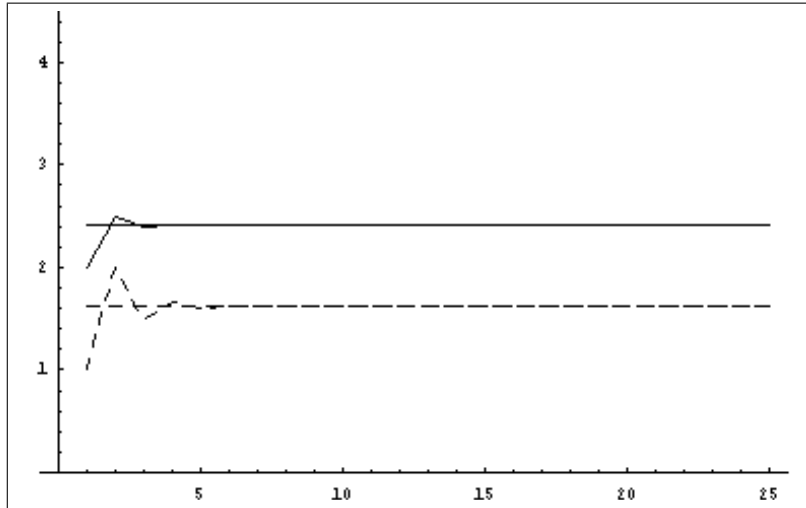

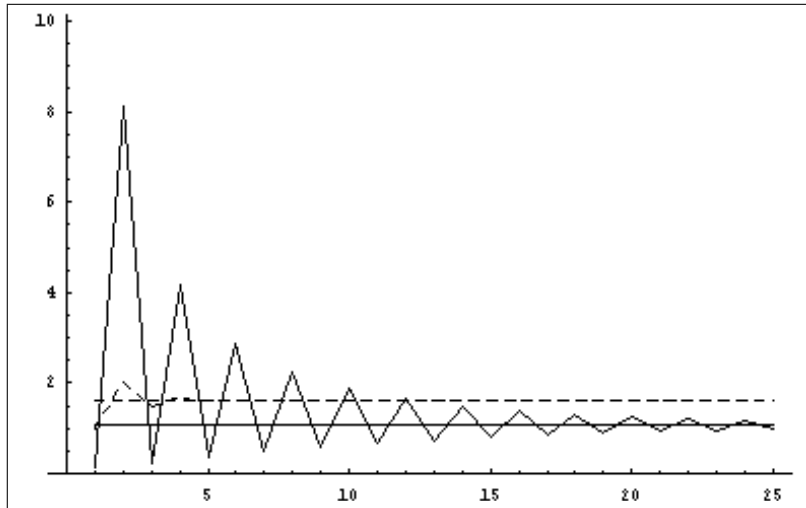Figure 1. $\mathrm{cf}_n(1)$ dashed line and $\mathrm{cf}_n(2)$ solid.

Figure 2. cf(1) dashed line and cf(1/8) solid.

But what about when $y < 1$? For example, if $y = 1/8$, we see in Figure 2 that the dashed line graph of $cf_n(1)$ converges much more rapidly than the solid line graph of $cf_n(1/8)$. Does this mean that $\phi_{1/8}$ which equals $(1 + \sqrt{257})/16$

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is more irrational than $(1 + \sqrt{5})/2$? No, it just means that $cf_n(1/8)$ converges more slowly than $cf_n(1)$.

Another proof that the continued fraction solution to Equation (1) converges more slowly for $y = 1$ than it does for $0 < y < 1$ would be to use a result proved by Euler in 1748 according to Wikipedia[2].

The relevant part of this theorem is as follows. If r_1 is the positive root, $\frac{y+\sqrt{y^2+4)}}{2}$ and r_2 is the negative root, $\frac{y-\sqrt{y^2+4)}}{2}$, then "... the rate of convergence of the continued fraction depends upon the ratio,$|r_2/r_1|$: the farther away that ratio is from unity, the more quickly the continued fraction converges." If we compute this ratio for values of y between 0 and 1, we see that their continued fractions converge more slowly than the one for ϕ.

But, if a continued fraction sequence converges more slowly, doesn't that make the number more irrational? Not necessarily, it just may be true that these continued fraction sequences are less efficient than some other algorithm that can be used to obtain one of the irrational ϕ_y's. Does it even make any sense to say one irrational number is more or less irrational than another? That is, does anyone have a satisfactory definition for comparing irrationalities? Is the algebraic number ϕ "more irrational" than the transcendental number π?

By the way, the nested root approximations to ϕ_y, defined in Equation (4), exhibit just the opposite rate of convergence. That is, if $y > 1$, then nested roots converge to ϕ_y more slowly than they do when $y = 1$. And if $0 < y < 1$, they converge more rapidly than they do when $y = 1$. So take your pick; which statement do you want to prove? Is ϕ the *least irrational* number or is it the *most irrational* number? Neither of these two questions make any sense.

2   References

1. M. Livio, The Golden Ratio: *The story of Phi, the World's Astonishing Number*, Broadway Books, 2002

2. Wikipedia, http://en.wikipedia.org/wiki/Solving_quadratic_equations_with_continued_fractions

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Send me your comments.