Generalizations of the Golden Ratio

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Introduction

Of the many interesting formulas and identities related to the Fibonacci sequence, which ones are also true for other second order recurrence equations? More pointedly, we wonder whether or not there exists even just one identity associated with Fibonacci sequence or the golden ratio, ϕ, that is not also true in a more general setting.

What about the claim in Livio[4], that "among the irrational numbers, ϕ is most irrational."? This astonishing assertion has been made, based upon the fact that the continued fraction for ϕ, consisting of only 1's, converges more slowly than the continued fraction of any other irrational number. Is this true? In our discussion of a generalized golden ratio, we will examine convergence of continued fractions regarding this question.

In this paper we state a definition for a Generalized Fibonacci Sequence (GFS) and the generalization of several important identities, such as Binet's formula, Cassini's identity, and others. We also lay out a path for the readers to develop their own proofs.

Let us begin by showing that for each real number $y > 0$, there is a y-Fibonacci sequence that yields a golden ratio ϕ_y whose properties (at least the one's we've checked) are identical to the those of ϕ when $y = 1$.

These ϕ_y's also occur in geometry. We will show how the diagonals of odd polygons are related to the generalized golden ratios. For other geometric results such as generalizations of various ratios and extreme mean properties in line segments, see Falbo[1], Fowler[2], and Steinbach [6].

A Generalized Fibonacci Sequence

Let y be any real positive number, and define a Generalized Fibonacci Sequence (GFS) as follows: A sequence $\{G_n(y)\}_{n=0}^{\infty}$, is called a GFS if and only if $G_0(y) = 0$, $G_1(y) = 1$, and for all $n \geq 0$

$$G_{n+2}(y) = yG_{n+1}(y) + G_n(y).$$

For brevity, use G_n in place of $G_n(y)$, so we can write

$$G_{n+2} = yG_{n+1} + G_n. \tag{1}$$

Thus, for $y > 0$, the first few terms of this GFS are:

$$0,\ 1,\ y,\ y^2+1,\ y^3+2y,\ y^4+3y^2+1,\ ....$$

Golden Ratio
The following limit holds

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \frac{y+\sqrt{y^2+4}}{2}. \tag{2}$$

This is proved by the method of characteristics. Treat (1) as a second order recurrence equation, and find x such that, as $n \to \infty$, $G_n \to x^n$. In other words, solve $x^{n+2} = yx^{n+1} + x^n$, for x. This gives us the quadratic equation

$$x^2 - yx - 1 = 0. \tag{3}$$

The limit shown in (2) is the positive root of Equation (3), denoted by ϕ_y, which we will call the "generalized golden ratio"

$$\phi_y = \frac{y+\sqrt{y^2+4}}{2}.$$

The other root to Equation (3) may be written as

$$\frac{-1}{\phi_y} = \frac{y-\sqrt{y^2+4}}{2}.$$

Two interesting special cases of ϕ_y are ϕ_2, and $\phi_{7/12}$. The first $\phi_2 = 1+\sqrt{2}$, sometimes called the "silver ratio", is the limit of the ratio of consecutive terms of the Pell numbers: $0, 1, 5, 12, 29, ...$ which is the GFS with $y = 2$. The second of these, is the rational number $\frac{4}{3}$. That is, if $y = 7/12$, then the $\lim_{n\to\infty}(G_{n+1}/G_n) = \phi_{7/12} = \frac{4}{3}$. What makes this interesting is that $\frac{4}{3}$ is closer to the *actual* proportion found in the spiral approximation to the shape of the *Nautilus pompilius* shell, as shown in Falbo[1], contrary to a popular myth that ϕ_1 describes this shape.

Binet's Formula
Using the roots of Equation (3), we can find coefficients A and B such that the general term of the GFS can be written as

$$G_n(y) = A\left(\phi_y\right)^n + B(-1/\phi_y)^n. \tag{4}$$

To determine these coefficients, look at the two initial values $G_0 = 0$ and $G_1 = 1$, thus

$$\begin{aligned} A + B &= 0, \text{ and} \\ A\phi_y + B\left(-1/\phi_y\right) &= 1. \end{aligned}$$

So, $A = (1/\sqrt{y^2+4})$, and $B = (-1/\sqrt{y^2+4})$. These give us the general solution to the recurrence Equation (1)

$$G_n = \frac{\left(\phi_y\right)^n - \left(-1/\phi_y\right)^n}{\sqrt{y^2+4}}. \tag{5}$$

In the ordinary Fibonacci sequence, the general solution to $F_{n+2} = F_{n+1} + F_n$ is

$$F_n = \frac{(\phi)^n - (-1/\phi)^n}{\sqrt{5}}, \tag{6}$$

a special case of Equation (5). Equation (6) is called Binet's formula. Equation (5) is a generalization of this formula for the GFS.

Elementary Formulas

In order to get simple proofs for other GFS formulas we re-state Equation (3) as follows.

$$(\phi_y)^2 - y\phi_y - 1 = 0. \tag{7}$$

Thus, we have

$$\phi_y = \sqrt{1 + y\phi_y}, \tag{7a}$$

$$y = \phi_y - \frac{1}{\phi_y}, \text{ and} \tag{7b}$$

$$\sqrt{y^2+4} = \phi_y + \frac{1}{\phi_y}. \tag{7c}$$

Equations (7b) and (7c) express the sum and difference of the roots. Equations (7a) and (7b) can be used to prove, for any positive number y, the following continued fraction and the nested roots equations

$$\phi_y = y + \cfrac{1}{y + \cfrac{1}{y + \cfrac{1}{y+1} \cdots}}, \tag{8}$$

$$\phi_y = \sqrt{1 + y\sqrt{1 + y\sqrt{1 + y\sqrt{1 + \ldots}}}} \tag{9}$$

The continued fraction does not prove that ϕ is the "most irrational number."

Why is the continued fraction for phi not a proof that "phi is the most irrational number?" When $y > 1$, the continued fraction in Equation(8), converges more rapidly than it would were $y = 1$. For this reason, the continued fraction for $y = 1$ is said to produce an irrational number that is the "most irrational number." See Livio[4]. Unfortunately, if $0 < y < 1$ the continued fraction for ϕ_y converges more slowly than for ϕ_1. We can illustrate the various rates of convergence by graphs. (I owe thanks to my wife Jean for giving me this idea.) To be precise, let us define a sequence of continued fractions for a given y as follows

$$\begin{aligned} cf_1(y) &= y \\ cf_n(y) &= y + \frac{1}{cf_{n-1}(y)}, \end{aligned}$$

where n is a positive integer, and $n > 1$.

Thus, when $y = 1$, $\{cf_n(1)\}_{n=1}^{\infty}$ is the continued fraction for ϕ,

$$\lim_{n\to\infty} cf_n(1) = \phi.$$

Does $cf_n(1)$ converge more slowly than $cf_n(2)$? Yes, as can be seen in Figure 1, the dashed line graph shows that it takes more terms of $cf_n(1)$ to settle down onto ϕ_1, than it does for the graph of $cf_n(2)$ to settle onto ϕ_2.

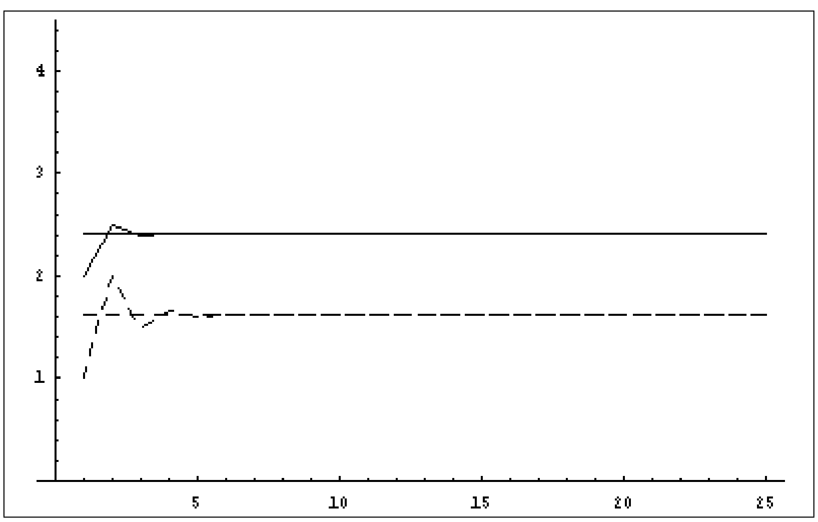

Figure 1. $\mathrm{cf}_n(1)$ dashed line and $\mathrm{cf}_n(2)$ solid.

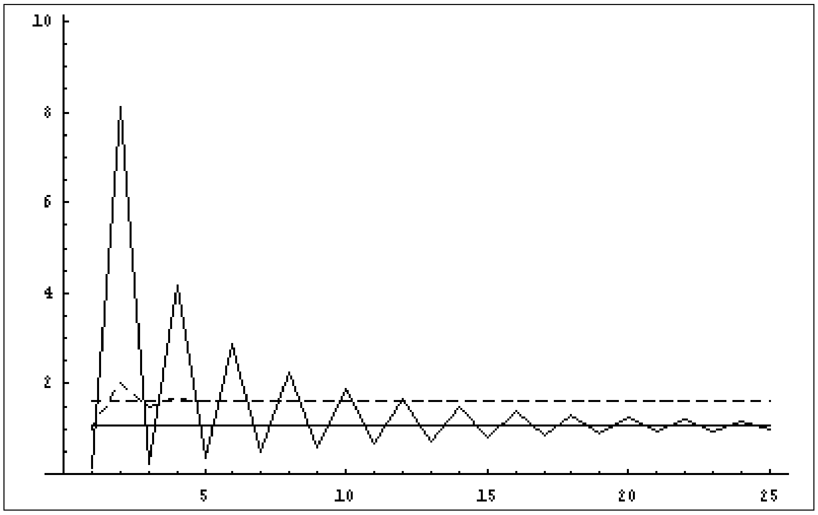

Figure 2. cf(1) dashed line and cf(1/8) solid.

But what about when $y < 1$? For example, if $y = 1/8$, we see in Figure 2 that the dashed line graph of $cf_n(1)$ converges much more rapidly than the solid line graph of $cf_n(1/8)$. Does this mean that $\phi_{1/8}$ which equals $(1+\sqrt{257})/16$ is more irrational than $(1+\sqrt{5})/2$? No, it just means that $cf_n(1/8)$ converges more slowly than $cf_n(1)$.

But, if a continued fraction sequence converges more slowly, doesn't that make the number more irrational? Not necessarily, it just may be true that these continued fraction sequences are less efficient than some other algorithm that can be used to obtain one of the irrational ϕ_y's. Does it even make any sense to say one irrational number is more or less irrational than another? That is, does anyone have a satisfactory definition for comparing irrationalities? Is the algebraic number ϕ "more irrational" than the transcendental number π?

By the way, the nested root approximations to ϕ_y, defined in Equation (9), exhibit just the opposite rate of convergence. That is, if $y > 1$, then nested roots converge to ϕ_y more slowly than they do when $y = 1$. And if $0 < y < 1$, they converge more rapidly than they do when $y = 1$. So take your pick; which statement do you want to prove? Is ϕ the *least irrational* number or is it the *most irrational* number? Neither of these two questions make any sense.

Formulas relating ϕ to the nth term of the Fibonacci sequence

In addition to Binet's formula given in Equation (6), above, for any integer $n \geq 1$, there are several well established formulas relating the *nth* term of the Fibonacci sequence to the golden ratio ϕ Two of these are

$$\phi^n = F_n\phi + F_{n-1}, \text{ and} \tag{10}$$

$$\phi^n = \frac{(F_{n-1} + \sqrt{5}F_n + F_{n+1})}{2}. \tag{11}$$

We will now state and indicate methods of proofs for generalizations of these formulas, and others, for the GFS.

Formulas relating ϕ_y to the nth term of the GFS

In addition to the generalization of Binet's formula given in Equation (5), the following two equations are generalizations of Equations (10) and (11) above.

$$(\phi_y)^n = G_n\phi_y + G_{n-1}, \text{ and} \tag{12}$$

$$(\phi_y)^n = \frac{G_{n-1} + \sqrt{y^2+4}G_n + G_{n+1}}{2}. \tag{13}$$

Equation (12) can be proved by induction. Equation (13) directly follows from Equation (12) and the definition of G_n. We leave the proof of Equation (12) to the reader; to prove (13), start with

$$(\phi_y)^n = G_n\phi_y + G_{n-1},$$

and write out ϕ_y on the right-hand side.

$$\begin{aligned}(\phi_y)^n &= G_n(\frac{y+\sqrt{y^2+4}}{2}) + G_{n-1} \\ &= \frac{G_n y + G_{n-1} + G_{n-1} + G_n\sqrt{y^2+4}}{2}.\end{aligned}$$

from which, equation (13) follows.

Cassini's Identity

If $n \geq 1$, in the Fibonacci sequence the equation

$$F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n,$$

is called Cassini's identity.

In the GFS we can prove a generalized Cassini,

$$G_{n-1}G_{n+1} - (G_n)^2 = (-1)^n. \tag{14}$$

Starting with Equation (5) for $n-1, n+1$ and n, we get

$$G_{n-1}G_{n+1} = \frac{(\phi_y)^{n-1} - (-1/\phi_y)^{n-1}}{\sqrt{y^2+4}} \cdot \frac{(\phi_y)^{n+1} - (-1/\phi_y)^{n+1}}{\sqrt{y^2+4}},$$

and

$$(G_n)^2 = \frac{(\phi_y)^{2n} - 2(-1)^n + (\phi_y)^{-2n}}{y^2+4}.$$

Their difference reduces to

$$G_{n-1}G_{n+1} - (G_n)^2 = \frac{(-1)^n \left(\phi_y - (-1/\phi_y)\right)^2}{y^2+4};$$

now using Equation (7c), we get Equation (14).

In equation (15) shown below, we demonstrate another generalization of Cassini's coming from a different GFS, $\{G'_n\}_{n=0}^{\infty}$. Starting with the two initial values $G'_0 = 1$ and $G'_1 = 1$ (instead of 0 and 1) and for $n \geq 0$, $G'_{n+2} = yG'_{n+1} + G'_n$. In this case, the Cassini-like formula is

$$G'_{n-1}G'_{n+1} - (G'_n)^2 = (-1)^{n+1}y. \tag{15}$$

Askey's Identity

In Osler and Hilburn[5], Askey's hyperbolic sine identity for the ordinary Fibonacci sequence is given as

$$F_m = \frac{2}{i^m\sqrt{5}} \sinh(m \log(i\phi)). \tag{16}$$

We can generalize this for the GFS to read

$$G_m = \frac{2}{i^m\sqrt{y^2+4}} \sinh(m \log(i\phi_y)). \tag{17}$$

To prove Equation (17), we start by writing the exponential version of the hyperbolic sine,

$$\begin{aligned}
\frac{2}{i^m\sqrt{y^2+4}}\sinh(m\log(i\phi_y)) &= \frac{2}{i^m\sqrt{y^2+4}}\frac{e^{m\log(i\phi_y)}-e^{-m\log(i\phi_y)}}{2}, \\
&= \frac{e^{\log(i^m\phi_y^m)}-\ e^{\log(i^{-m}\phi_y^{-m})}}{i^m\sqrt{y^2+4}}, \\
&= \frac{i^m\phi_y^m-i^{-m}\phi_y^{-m}}{i^m\sqrt{y^2+4}}, \\
&= \frac{\phi_y^m-(-\phi_y)^{-m}}{\sqrt{y^2+4}}.
\end{aligned}$$

Then, by equation (5), we get Equation (17).

Catalan's Identity

Another property of the ordinary Fibonacci sequence is Catalan's identity,

$$(F_n)^2 - F_{n-r}F_{n+r} = (-1)^{n-r}(F_r)^2,$$

where $n \geq 1$ and $r \leq n$. The following generalization of this can be proved by Equation (5),

$$(G_n)^2 - G_{n-r}G_{n+r} = (-1)^{n-r}\left(G_r\right)^2. \tag{18}$$

Write $(G_n)^2$ and G_n, G_{n-r} as

$$\begin{aligned}
(G_n)^2 &= \frac{(\phi_y)^{2n}-2(-1)^n+(1/\phi_y)^{2n}}{y^2+4}, \text{ and} \\
G_{n-r}G_{n+r} &= \frac{(\phi_y)^{2n}+(1/\phi_y)^{2n}-(-1)^{n-r}[(\phi_y)^{2r}+(1/\phi_y)^{2r}]}{y^2+4}.
\end{aligned}$$

Then, their difference is

$$\frac{-2(-1)^n+(-1)^{n-r}[(\phi_y)^{2r}+(1/\phi_y)^{2r}]}{y^2+4}$$

or

$$(-1)^{n-r}\frac{[(\phi_y)^{2r}+(-1/\phi_y)^{2r}-2(-1)^r]}{y^2+4}$$

$$(-1)^{n-r}\left(\frac{[(\phi_y)^r-(-1/\phi_y)^r]}{\sqrt{y^2+4}}\right)^2.$$

This proves Equation (18). Here we were able to factor out $(-1)^{n-r}$ from $(-1)^n$, because for any integers n and r, $(-1)^{n-r}(-1)^n = (-1)^{-r} = (-1)^r$.

d'Ocagne's Identity

The equation

$$F_m F_{n+1} - F_n F_{m+1} = (-1)^n G_{m-n},$$

called d'Ocagne's identity can be generalized to read

$$G_m G_{n+1} - G_n G_{m+1} = (-1)^n G_{m-n}, \tag{19}$$

and proved by Equation (5) with the help of Equation (7c). Here is how. Use the generalized Binet's formula, Equation (5), on each of G_m, G_{n+1}, G_n and $G_{m+1.}$ Then

$$\begin{aligned} G_m G_{n+1} &= \frac{\left((\phi_y)^m - (-1/\phi_y)^m\right)\left((\phi_y)^{n+1} - (-1/\phi_y)^{n+1}\right)}{y^2+4}, \text{ and} \\ G_n G_{m+1} &= \frac{\left((\phi_y)^n - (-1/\phi_y)^n\right)\left((\phi_y)^{m+1} - (-1/\phi_y)^{m+1}\right)}{y^2+4}. \end{aligned}$$

Compute their difference which, eventually, simplifies to

$$G_m G_{n+1} - G_n G_{m+1} = (-1)^n \frac{\left((\phi_y)^{m-n} - (-1)^{m-n}\phi_y^{n-m}\right)(\phi_y + 1/\phi_y)}{y^2+4}.$$

Now, using equation (7c), we get equation (19).

Finding ϕ_y in odd polygons

It has already been shown, elsewhere, that geometric properties of ϕ_1(including the "extreme mean" properties) are also shared by the generalized golden ratios. See Falbo [1] and Fowler [2], and Steinbach[6]. In addition, Falbo[1] has indicated that a generalized golden ratio exists for every regular odd polygon, beyond the pentagon. That is, if we let $n \geq 2$ be a positive integer, x be the length of a longest diagonal, and y be the length of the second longest diagonal in any regular $(2n+1)$-sided polygon with unit sides, then $x - y = 1/x$. The values of x and y work out, in general, to be

$$y = \frac{1 - 4\cos^2(\frac{n\pi}{2n+1})}{2\cos(\frac{n\pi}{2n+1})}, \tag{20}$$

and

$$x = \frac{1}{2\cos(\frac{n\pi}{2n+1})}. \tag{21}$$

Since $x - y = 1/x$, then x is the positive solution of the quadratic equation $x^2 - yx - 1 = 0$, that is

$$x = \phi_y = \frac{y + \sqrt{y^2 + 4}}{2}.$$

A proof of Equations (20) and (21) can be obtained by trigonometry; specifically, the law of cosines is useful in the derivation. Note that, in the regular unit pentagon, a second longest "diagonal", $y = 1$ is actually one of the sides. Thus, $x = \phi$ in the pentagon.

We will use a heptagon to prove Equations (20) and (21) for $n = 3$. The general case can be similarly proved. Consider a regular heptagon $ABCDEFG$, with unit sides. The diagonal $\overline{AE}$ is a longest diagonal. Let x be the length of $\overline{AE}$. The diagonal $\overline{AF}$ is a second longest diagonal; let its length be y. Let $\overline{AM}$ be the perpendicular bisector of the side $\overline{ED}$.

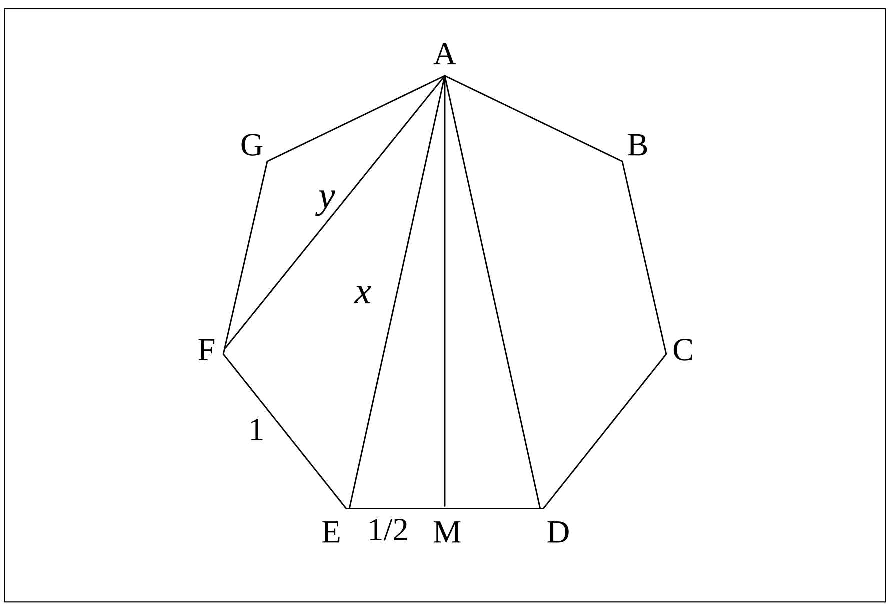

Figure 3. Diagonals of lengths x and y of a heptagon

As the reader can verify, the angle measures are as follows:

$$\begin{aligned}
\measuredangle EAD &= \frac{\pi}{7} = \measuredangle EAF \\
\measuredangle AED &= \frac{3\pi}{7} = \measuredangle ADE \\
\measuredangle AEF &= \frac{2\pi}{7} \\
\measuredangle AFE &= \frac{4\pi}{7}
\end{aligned}$$

So, now we have

$$\begin{aligned}
\cos(\frac{3\pi}{7}) &= \frac{1/2}{x}, \text{ or} \\
x &= \frac{1}{2\cos(3\pi/7)},
\end{aligned}$$

which is Equation (21) for $n = 3$.

Now to prove Equation (20), we use the law of cosines

$$x^2 = 1 + y^2 - 2y\cos(\frac{4\pi}{7}). \tag{22}$$

Next, note that $\cos(\frac{4\pi}{7}) = -\cos(\frac{3\pi}{7})$, so that Equation (22) becomes

$$x^2 = 1 + y^2 + 2y\cos(\frac{3\pi}{7}). \tag{23}$$

But $2\cos(\frac{3\pi}{7}) = \frac{1}{x}$, therefore we may re-write Equation (23) as

$$\begin{aligned}
x^2 - y^2 &= 1 + \frac{y}{x}, \\
(x-y)(x+y) &= \frac{x+y}{x}, \\
y &= x - \frac{1}{x}.
\end{aligned}$$

which yields Equation (20), when $n = 3$.

Other questions

An interesting area for further studies would be to investigate the full two term sequence $\{H_n(y,z)\}_{n=0}^{\infty}$, where y and z are any real numbers, $H_0(y,z) =$

0, $H_1(y, z) = 1$, and $H_n(y, z) = yH_{n-1}(y, z) + zH_{n-2}(y, z)$, for $n \geq 2$, and to determine what form is to be taken by the generalization of Binet's formula, as well as the other formulas. An excellent attempt to collect some Fibonacci formulas and to generalize them into such two-term recurrence sequences can be found in Kalman and Mena[3].

Summary

What we have shown here is that ϕ (or in our notation, ϕ_1) is but one of an uncountable infinity of numbers, ϕ_y, all exhibiting the algebraic properties listed here. The GFS allows us to state generalized versions of continued fractions, Binet's formula and several other formulas. We also discussed how generalized golden ratios occur in geometry and gave a formula for finding these as diagonals in regular odd polygons with 5 or more sides.

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