# Some Elementary Methods for Solving Functional Differential Equations

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## Abstract

This paper is an introduction, for non-specialists, to effective ways for solving most Delay Differential Equations(DDEs) and some other, more general, Functional Differential Equations (FDEs). It is a novel synthesis of elementary methods that use only the basic techniques taught in a first course of Ordinary Differential Equations.

DDEs and FDEs are often used as modeling tools in several areas of applied mathematics, including the study of epidemics, age-structured population growth, automation, traffic flow and problems related to the engineering of high-rise buildings for earthquake protection.

We discuss the solution of constant coefficient-DDEs by the "Method of Characteristics," and we show how to solve more general DDEs using Myshkis' "Method of Steps." This method is one of those "natural" procedures that are often repeatedly discovered by workers in the field; it is easy to understand and use. We also discuss differentiation as a way to solve certain special types of "reverse time" FDEs that have interesting applications in biology.

# 1 Delay Differential Equations

DDEs are well known to be useful in various fields such as age-structured population growth, control theory, and any model involving responses with non-zero delays; these include models of conveyor belts, urban traffic, heat exchangers, robotics, and chatter. Chatter is described by Asi and Ulsoy [1] as "...a self-excited vibration, which is the result of an interaction between the tool structure and the cutting process dynamics." Driver *et al.* [5], discuss mixture problems that do not assume instantaneously perfectly mixed solutions. Applications to delays in transportation, signal transmission, genetic repression, control systems for nuclear reactors with time delays, and other diverse topics may be found in Hale and Lunel [9], and in Wu [17].

The general first order DDE has the form: $y'(t) = f(t, y(t), y(t-d))$, for some given $d > 0$. Here, $y'(t)$ depends on the value of $y$ at some time $t-d$ in the past, as well as depending upon the current value of $y$ , and other functions of $t$ as determined by $f$. Applications are discussed in the following additional references: Bender and Neuwirth[3], Insperger and Stepan[11], Nesbit[14], Asi[1].

## 1.1 Linear DDEs

The most fundamental Functional Differential Equation (FDE) is the linear first order Delay Differential Equation,

$$y'(t) = a_1(t)y(t) + a_2(t)y(t-d), \text{ for } t \geq 0. \tag{1}$$

Equation (1) is usually accompanied by an auxiliary condition, stated in terms of a function revealing the state of the system for a period prior to the initial time, $t = 0$. The auxilliary function is sometimes called the "history" function for the system, but as we shall see later, it is really a "remote control" function, describing the behavior of $y$ on a remote time-interval other than the one upon which the differential equation is defined. For a complete statement of the first order delay system, let $d > 0$, and $a_1, a_2$ be class $\mathcal{C}^1$ functions on $[0, d]$ and let $p(t)$ be a class $\mathcal{C}^1$ function on $[-d, 0]$, then there is a unique function $y(t)$ satisfying the system:

$$y'(t) = a_1(t)y(t) + a_2(t)y(t-d), \text{ for } t \in [0, d] \quad (2)$$

$$y(t) = p(t), \text{ for } t \in [-d, 0] \quad (3)$$

It is quite easy to prove that the system (2),(3) cannot have more than one solution. Simply assume two solutions $u(t)$ and $y(t)$, and let $g(t)$ be their difference $u(t) - y(t)$. Then $g(t)$ satisfies Equation (2), $g'(t) = a_1(t)g(t) + a_2(t)g(t-d)$,on $[0, d]$. But since $u$ and $y$ both satisfy the same auxilliary equation, (3) becomes $g(t) \equiv 0$ for $t \in [-d, 0]$,making $g(t-d) \equiv 0$ for $t \in [0, d]$. The resulting first order ODE system $g'(t) = a_1(t)g(t)$ on $[0, d]$, with $g[0] = 0$ proves $g(t) \equiv 0$ on $[0, d]$,as well as on $[-d, 0]$, thus $u(t) \equiv y(t)$ on $[-d, d]$.

The system (2),(3) has a solution, which we show by actually getting one. Of the several methods used for solving this system, we will discuss two – both of which can be taught in a first or second course of differential equations or in a course in numerical analysis.

The Method of Characteristics (MOC)
The Myshkys Method of Steps (STEPS)

Turning first to the MOC, we find it is suitable for solving the simplest case of equation (2), namely the constant-coefficient equation

$$y'(t) = a_1 y(t) + a_2 y(t-d) \quad (4)$$

with both $a_1$ and $a_2$ constant. One of the beneficial consequence of using the MOC is that we are introduced to an interesing function called the Lambert w-function, $W(z)$, namely the inverse of the equation

$$z(w) = we^w \quad (5)$$

### 1.1.1 Method of Characteristics

We will assume that the solution to (4) has the form of $y(t) = e^{mt}$, for some constant $m$, (real or complex). So, we set $y(t) = Ce^{mt}$, with $C$ arbitrary, getting

$$Cme^{mt} = a_1 Ce^{mt} + a_2 Ce^{mt-md} \quad (6)$$

Division by $Ce^{mt}$ reduces (6) to the characteristic equation

$$(m - a_1)e^{md} - a_2 = 0 \tag{7}$$

Notice that when $a_2 = 0$, $m = a_1$, giving us $y(t) = Ce^{a_1 t}$, the solution to the ODE $y'(t) = a_1 y(t)$, which is Equation (4), with $a_2 = 0$. On the other hand, in the pure delay equation, where $a_1 = 0$, but $a_2 \neq 0$, the characteristic equation becomes

$$me^{md} = a_2$$

Multiplying by $d$ converts this to the inverse of the Lambert function

$$mde^{md} = a_2 d \tag{8}$$

That is, $md = W(a_2 d)$, thus $m = \frac{1}{d}W(a_2 d)$. $W(a_2 d)$ has either no, one or two real roots depending upon whether $a_2$ is negative and $<, =,$ or $> \frac{-1}{de}$ and it has only one real root if $a_2 > 0$. Under any of these conditions $W(a_2 d)$ has infinitely many complex roots, $r_k + is_k$.

Thus, the MOC produces the following solution to Equation (4) with $a_1 = 0$.

$$y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} + \sum_{k=1}^{\infty} e^{r_k t} \left[ c_{(1,k)} \cos(s_k t) + c_{(2,k)} \sin(s_k t) \right] \tag{9}$$

with $c_1$ or $c_2$ zero or not zero according to the above restrictions on $a_2$ and $\frac{-1}{de}$. For a derivation of this, see my paper, Falbo[6].

In order to determine the coefficients $c_1, c_2, c_{(1,k)}, c_{(2,k)}$, we could use the history function $p(t)$ and its derivatives, or we could use $p(t)$ to create a least squares fit as suggested by Hefferman[10].

To solve Equation (4) where both $a_1$ and $a_2$ are not zero,we simply notice that a change in variables in Equation (7) can be used to reduce the equation to the form of Equation (8). *Viz*, let $m - a_1 = n$, then from Equation (7), we get

$$ne^{nd+da_1} - a_2 = 0$$

So,

$$ne^{nd} = a_2 e^{-da_1}$$

Multiply through by $d$

$$nde^{nd} = a_2 de^{-a_1 d} \tag{10}$$

Using $nd = W(a_2 de^{-a_1 d})$, we get the real and complex roots of Equation (10). Divide by $d$ to get $n$, then add $a_1$ to get the real and complex values $m$ satisfying the characteristic equaton (7).

It should be noted that the MOC can be applied to a higher order constant coefficient linear DDE. This is done in the usual way of writing a linear nth order differential equation as a system of n first order equations.

### 1.1.2 Method of Steps

The method of steps is much more intuitive and can be used to solve DDEs with variable coefficients. Although this method may have been discovered many times by several workers, we will cite as our reference, Myshkis [13] from the Soviet Encyclopeadia of Mathematics.

This method converts the DDE on a given interval to an ODE over that interval, by using the known history function for that interval. The resulting equation is solved, and the process is repeated in the next inerval with the newly found solution serving as the history function for the next interval. We will show how to apply it to the system of equations (2) and (3).

**Step 1** On the interval $[-d, 0]$, the function $y(t)$ is the given function $p(t)$, so $y(t)$ is known there. Thus, we say the equation is "solved" for the interval $[-d, 0]$, call this solution $y_0(t)$.

**Note 1:** When $t \in [0, d]$, $t - d \in [-d, 0]$, so $y(t - d)$ becomes $y_0(t - d)$ on $[0, d]$.

**Step 2** In the interval $[0, d]$, the system (2),(3) becomes

$$\begin{aligned} y'(t) &= a_1(t)y(t) + a_2(t)y_0(t-d), \text{ on } [0, d] \\ y(0) &= p(0) \end{aligned} \tag{11}$$

Equation (11) is an ODE and *not* a delay equation because $y_0(t - d)$ is *known*; it is simply $p(t - d)$. Thus, we solve this ODE on $[0, d]$, using $y(0) = p(0)$ as our initial condition. Denote by $y_1(t)$ this solution on the interval $[0, d]$.

**Note 2:** Solving Equation (11) may be accomplished by treating it as the nonhomogeneous equation,

$$\begin{aligned} y'(t) - a_1(t)y(t) &= a_2(t)p(t-d), \text{on } [0, d] \\ y(0) &= p(0) \end{aligned}$$

and using techniques taught in an early course in differential equatons, such as an Integrating Factor $IF = e^{-\int a_1(t)dt}$, for a closed form solution, or by numerical methods, for an approximate solution.

**Step 3** On the interval $[d, 2d]$, the system becomes

$$\begin{aligned} y'(t) &= a_1(t)y(t) + a_2(t)y_1(t-d), \text{ on } [d, 2d] \\ y(d) &= y_1(d) \end{aligned} \tag{12}$$

which is again an ODE. We solve this, using the initial condition at $d$ and get a solution $y_2(t)$ for our system on $[d, 2d]$. These steps may be continued for subsequent intervals.

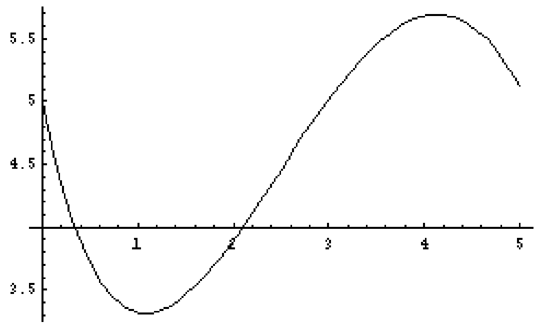


Figure 1: Solution to (13)

*EXAMPLE 1*

Problem. Find one step of the solution to the system (2)-(3) for the following data

$d = 5$

$p(t) = d - t(t + d)$

$a_1 = -1$

$a_2 = 0.5.$

Solution. Assigning these values, we get

$$\begin{aligned} y'(t) &= -y(t) + 0.5y(t-d), \text{ on } [0,5] \\ y(t) &= d - t(t+d) \text{ on } [-5,0] \end{aligned} \tag{13}$$

Now replace $y(t-d)$ by $p(t-d)$, and use the history function to get the initial condition $y(0) = p(0)$. On the first interval the solution $y_1(t)$ will be the function satisfying

$$y'(t) = -y(t) + 0.5p(t-d)$$
$$y(0) = p(0)$$

Whose solution we can obtain directly or numerically; its graph is shown in Figure 1.

The error graph for this numerical solution is given in Figure 2.

## 1.2 The General First Order DDE

The general (linear or nonlinear) first order delay differential equation can be written

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-d)) \text{ for } t \in [0,d] \\ y(t) &= p(t), \text{ for } t \in [-d, 0] \end{aligned} \tag{14}$$

with appropriate conditons for $f$ and $p$. Just as in the linear cases described in he preceding section, STEPS works here as well.

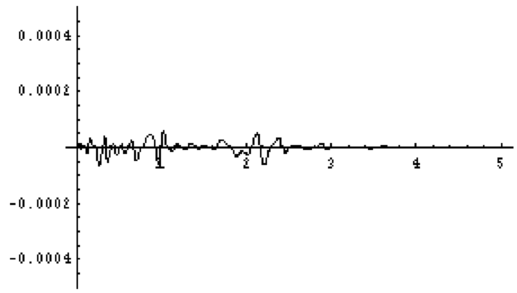


Figure 2: Error for Figure 1

**Step 1.** On the interval $[-d, 0]$ the function $y(t)$ is the given function $p(t)$; this is $y_0(t)$.

**Step 2.** In the interval $[0, d]$, the system (14) becomes

$$\begin{aligned} y'(t) &= f(t, y(t), y_0(t-d)) \text{ on } [0, d] \\ y(0) &= p(0) \end{aligned} \tag{15}$$

Equation 15 is an ODE and *not* a delay equation because $y_0(t-d)$ is known; it is simply $p(t-d)$ for $t \in [0, d]$. Solving (15), we obtain a solution on $[0, d]$, call it $y_1(t)$. On the next interval,$[d, 2d]$, we solve $y'(t) = f(t, y(t), y_1(t-d))$, with $y(d) = y_1(d)$, etc.

Here is why STEPS works in these nonlinear DDEs. The equation is defined on some given domain, $[0, d]$ but the expression $y(t-d)$ is reading its values from another domain, $[-d, 0]$ And on $[0, d]$, $y(t-d)$ is known because for all $t$ in $[-d, 0]$, $y(t) = p(t)$.. Therefore in $[0, d]$, we can replace $y(t-d)$ with $p(t-d)$. The process then can be continued for the next step. This is because the solution on $[0, d]$ becomes the known function that replaces $y(t-d)$ in the next interval.

The same thing happens anytime the current differential equation contains a term in which the "unknown" function $y$ is given for values in some other domain. In such a case, the expression for $y$ on the remote domain becomes a known function. Such a remote domain need not be the interval immediately preceding the current one. In which case it may not be possible to compute a second step. Nevertheless we can apply STEPS, at least once, to many FDE that are not just DDEs. We will discuss some of these in the next section.

## 2 Remote Control Dynamical Systems

If you send a signal to a robot telling it to turn, stop, or perform some other task, there will be some lag between the time you initiate the signal and the time the robot responds. It takes another delay for you to see what the robot did and then to make use of this feedback to influence your next decision about what new signal to send.

For another example, if you are trying to row a boat you may push your oar through the water and then wait to see the heading of the boat before dipping the oar again. However, if you are heading for a dangerous obstacle, you may not wait after each stroke, but simply decide to execute a series of pre-planned back strokes before getting feedback.

Typically, controls are not sent as indiviudal signals, one at a time, but rather as a pre-set pattern, a template. Almost any "automated" process works from a template. For instance, a pre-determined design can be programmed into a weaving machine to produce a desired pattern in a rug.

The methods we have already discussed may be used to solve equations employed in modelling a dynamic process with a given pre-set template, or control function The control function need not be simply a *past* action, but can express a desired future goal or target. It may not even be related to the remote action in the usual sense of "time."

Specifically, we want to solve a Remote Control Dynamical System (RCDS) which is defined as one whose dynamical equation is the FDE

$$y'(t) = f(t, y(t), y(h(t))), \ t \in \mathcal{I} \tag{16}$$

where $\mathcal{I}$, is an open interval called the *operational interval.* The *deviating argument*, $h(t) \in \mathcal{C}^1[\mathcal{I}]$, is one whose range, $h[\mathcal{I}]$, (called the *remote domain* or *remote interval*) is disjoint from $\mathcal{I}$. Initially, the system is assumed to be subject to a *control function*, $p(t) \in \mathcal{C}^1[h[\mathcal{I}]]$ defined on the remote domain. Thus, the *output function,* $y(t)$, is the solution of Equation (16) on $\mathcal{I}$, and $y(t) = p(t)$ on $h[\mathcal{I}]$.

When $d > 0$ and the deviating argument $h$ is defined by $h(t) = t - d$ we get the DDE. The function $t - d$ is called the *retarded argument.* If $h(t) = t + d$ the equation is called an *advanced argument.* If, the derivative is also of the form $y'(t \pm d)$, the equation is called *neutral.* Otherwise, some possible deviating arguments are: accelerated delays $h(t) = mt \pm d$, where $m > 0$, or nonlinear delays, $h(t) = g(t) \pm d$, if $g(t) \in \mathcal{C}^1[\mathcal{I}]$ and $g(t) \pm d \notin \mathcal{I}$.

Several applications that involve first and second order dynamical systems using delay equations with feedback, can be found in Olgak *et al.* [15]. An interesting version of this type of feedback, balancing an inverted pendulum, is discussed in Atay [2]. Another application using acceleration feedback includes connecting two buildings together by a shock absorber in order to reduce earthquake damage, Christenson and Spencer [4].

*EXAMPLE 2*

Problem: Using the following data

$d = 5$

$a_1 = -1$

$a_2 = 0.5$

$h(t) = t + d$

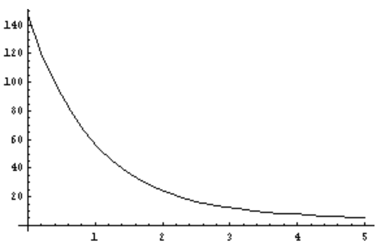


Figure 3: Solution to (17)

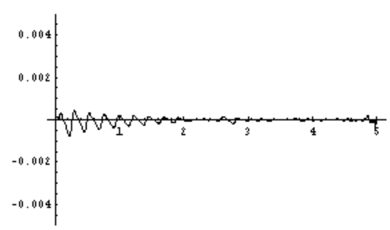


Figure 4: Error graph for the solution to (17)

$p(t) = d - (t-d)(t-2d)$

Solve on the interval $[0, d]$, the following remote control problem whch requires that the solution $y$ on $[0, d]$, take on the values of $p(t)$ in the future interval $[d, 2d]$.

$$\begin{array}{c} y'(t) = a_1 y(t) + a_2 y(h(t)), \text{ on } [0, d] \\ y(t) = p(t), \text{ on } [d, 2d] \end{array} \tag{17}$$

Solution: We use the first step of the method of STEPS, by replacing $y(h(t))$ on $[0, d]$ by its known value $p(h(t))$ on the interval $h[0, d]$ which is $[d, 2d]$, and we get

$$\begin{array}{c} y'(t) = a_1 y(t) + a_2 p(t + d) \\ y(d) = p(d) \end{array} \tag{18}$$

The graphical solution is shown in Figure 3.

A plot of the expression $y'(t) - a_1 y(t) - a_2 y(h(t))$ is shown in Figure 4. This represents the error in solving (17).

The following example shows that we can have an infinite remote domain.

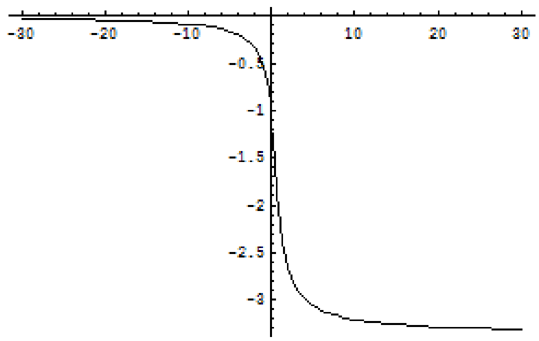


Figure 5: Solution to the system (19)

EXAMPLE 3

$$\begin{aligned} y'(t) &= 1.5y(-t^2) \text{ for all } t \in [0,\infty) \\ y(t) &= \tfrac{1}{t-1} \text{ for all } t \in (-\infty, 0] \end{aligned} \tag{19}$$

In Figure 5 the graphical solution to the system (19) is shown on $[-30, 30]$, not $(-\infty, \infty)$, but close enough!

### 2.0.1 A Time reversal Problem

In an experiment measuring the population growth of a species of water fleas, Nesbit [14], used a DDE model in his study. In simplified form his population equation was $N'(t) = a_1 N(t-d) + a_2 N(t)$. He ran into difficulty with this model because he did not have a reasonable history function to carry out the solution of this equation. To overcome this roadblock he proposed to solve a "time reversal" problem in which he sought the solution to an FDE that is neither a DDE, nor a RCDS. He used a "time reversal" equation to get the juvenile population prior to the beginning time $t = 0$. We express this as

$$\begin{aligned} p'(t) &= ap(-t) + bp(t) \\ p(0) &= c \end{aligned} \tag{20}$$

Where $c$ is the juvenile population at time $t = 0$. The equation also tells us that $p'(0) = (a+b)c$.

At first glance, it may seem that it is madness to try to solve such an equation. It turns out, however, that this equation can be solved and it can be shown to have a unique solution. The fundamental form of the solution is

$$p(t) = Ae^{rt} + Be^{-rt} \tag{21}$$

Where $r = \sqrt{b^2 - a^2}$, and is real or imaginary according to whether $b^2 \geq a^2$ or not. And the way we get this is by differentiating the equation!

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# 3 Idempotent Differential Equations

The time reversal problem in the preceding section is a special case of a type of equaton called an Idempotent Differential Equation, IDE. These are defined as equations of the form

$$\begin{aligned} y'(t) &= f(t, y(t), y(u(t)) \\ y(t_0) &= y_0 \end{aligned} \tag{22}$$

Where $u(t)$ is idempotent, that is $u(u(t)) = t$, and $t_0$ is a fixed point of $u$. For details see my paper, Falbo[7].

In Example 4, below, we consider the simplest IDE, one in which the deviating argument is $u(t) = d-t$. This function is idempotent since $u(u(t)) = u(d-t)$, which is $d - (d-t) = t$. Note $d - t$ is not the "delay" function $t - d$.

*EXAMPLE 4*

Problem:

If $d > 0$, and $a$ and $x_0$ are any real numbers, find a function $x(t) \in \mathcal{C}^\infty$ on some open interval $\mathcal{I}$ containing $d/2$ that satisfies the system

$$\begin{aligned} x'(t) &= ax(d-t) \\ x(d/2) &= x_0 \end{aligned} \tag{23}$$

Although Equation(23) is first order and linear, it has the following periodic function as its only solution.

$$x(t) = x_0 \cos(at - \frac{ad}{2}) + x_0 \sin(at - \frac{ad}{2}) \tag{24}$$

It is easy to show by substitution that (24) *is* a solution to the system (23), but how do we know it is the *only* solution? We will take the time to show that it has only one solution because in proving uniqueness, we will also introduce a method for getting the solution.

Suppose that there is a "second" solution $y(t) \in \mathcal{C}^\infty$ that satisfies (23) and its initial condition, then

$$\begin{aligned} y'(t) &= ay(d-t) \\ y(d/2) &= x_0 \end{aligned} \tag{25}$$

We can differentiate (25) getting

$$y''(t) = -ay'(d-t) \tag{26}$$

Now, for some positive number $\delta$, if $t \in [\frac{d}{2} - \delta, \frac{d}{2} + \delta]$, then $d - t$ is in this same interval. Thus, $y'(d-t)$ exists and by equation (25), it is equal to $ay(t)$. Substituting this into Equation (26) and noting that $y'(d/2) = ay(d - d/2) = ax_{0,}$ we get

$$\begin{aligned} y''(t) &= -a^2 y(t) \qquad (27) \\ y(d/2) &= x_0 \\ y'(d/2) &= ax_0 \end{aligned}$$

This is a second order ordinary differential equation and it has a unique solution, namely the one defined in Equation (24). Thus, this supposedly second solution $y(t)$ actually *is* $x(t)$. In other words, there is only one solution to both (23) and (27). When two different systems have the same unique solution like this, we will say the two systems are *equivalent* to each other.

We will now prove that every linear IDE can be solved by differentiation.

**Theorem 1** *Let $u(t)$ be a function that is its own inverse on an an open interval $\mathcal{I}$, $u \in \mathcal{C}^{\infty}[\mathcal{I}]$, and $t_0 \in \mathcal{I}$ be a fixed point of $u$. Let $y_0$ be any real number and each of $a(t), b(t)$ be a function of class $\mathcal{C}^{\infty}$ on $\mathcal{I}$, such that $a(t)$ does not vanish on the interval $\mathcal{I}$, then the system*

$$\begin{aligned} y'(t) &= a(t)y(u(t)) + b(t)y(t) \text{ on } \mathcal{I} \\ y(t_0) &= y_0 \end{aligned} \tag{28}$$

is equivalent to the ordinary second order differential equation system

$$\begin{aligned} y''(t) &= p(t)y'(t) + q(t)y(t) \text{ on } \mathcal{I} \\ y(t_0) &= y_0 \\ y'(t_0) &= (a(t_0) + b(t_0))y_0 \end{aligned} \tag{29}$$

Where

$$p(t) = \frac{a'(t)}{a(t)} + b(t) + b(u(t))u'(t) \tag{30}$$

$$q(t) = \frac{-a'(t)b(t)}{a(t)} + b'(t) + a(t)a(u(t))u'(t) - b(t)b(u(t)u'(t) \tag{31}$$

**Proof.** $\Longrightarrow$ By making use of the fact that $u(u(t)) = t$, we can easily show that if $y(t)$ is a solution to the system (28), then it must be a solution to (29). First, note that the range of $u$ is $\mathcal{I}$; therefore, for each $t \in \mathcal{I}$, $u(t)$ is in the domain of $a(t)$ and of $b(t)$. We obtain Equation (29) by differentiating Equation(28), getting

$$y''(t) = a(t)y'(u(t))u'(t) + a'(t)y(u(t)) + b(t)y'(t) + b'(t)y(t) \tag{32}$$

and substituting $u(t)$ for $t$ into Equation (28) gives us

$$\begin{aligned} y'(u(t)) &= a(u(t))y(u(u(t))) + b(u(t))y(u(t)) \text{ or} \\ y'(u(t)) &= a(u(t))y(t) + b(u(t))y(u(t)) \end{aligned} \tag{33}$$

Using equations (28), (32) and (33) we can eliminate the $y(u(t))$ and $y'(u(t))$ terms, yielding equations (30), (31) and (29). This proves that (28)$\Longrightarrow$(29) Now to prove $\Longleftarrow$ let $y(t)$ be the solution to Equation (29); we know it exists and is unique because the system (29) is a second order linear Ordinary Differential

Equation whose coefficients are in $\mathcal{C}^\infty[\mathcal{I}]$. Now, define the function $L(t)$, as follows

$$L(t) = y'(t) - a(t)y(u(t)) - b(t)y(t) \tag{34}$$

Where $u(t)$, $a(t)$, and $b(t)$ are as given in the hypothesis. $L(t)$ exists and is in $\mathcal{C}^\infty[\mathcal{I}]$. Note that $L(t_0) = 0$. We find $L'(t)$, and $L(u(t))$, and after a few computations, we will come out with

$$\begin{aligned} L'(t) &= A(t)L(u(t)) + B(t)L(t) \\ L(t_0) &= 0 \end{aligned} \tag{35}$$

where $A(t) = -a(t)u'(t)$, and $B(t) = a'(t)/a(t) + b(u(t))u'(t)$. The coefficients $A(t)$, and $B(t)$ and the initial condition in (29) are sufficient to establish that $L(t) \equiv 0$. Therefore, by (34) we have $y'(t) - a(t)y(u(t) - b(t)y(t) \equiv 0$. This shows that the function $y(t)$ found in (29) is the solution to (28). Thus, the two systems are equivalent. ■

*EXAMPLE 5*

In Equation (28), let $\mathcal{I}$ be the interval $(1, 5)$, $a(t) = t^2 + 1$, $b(t) = 0.5$, $y_0 = 1$, and $u(t) = 5/t$. Then the fixed point of $u$ is $t_0 = \sqrt{5}$. Thus, Equation (28) looks like this:

$$\begin{aligned} y'(t) &= (t^2 + 1)y(5/t) + 0.5y(t), t\epsilon(1, 5) \\ y(\sqrt{5}) &= 1 \end{aligned} \tag{36}$$

and from Equations (29) and (30), and (31) we are left with

$$\begin{aligned} y''(t) &= p(t)y'(t) + q(t)y(t), t\epsilon(1, 5) \\ p(t) &= (t^4/2 + 2t^3 - 2t^2 - 5/2)/(t^4 + t^2) \\ q(t) &= -(5t^6 + t^5 + 143.75t^4 + 253.75t^2 + 125)/(t^6 + t^2) \\ y(\sqrt{5}) &= 1 \\ y'(\sqrt{5}) &= 6.5 \end{aligned} \tag{37}$$

which is not solvable in closed form. We employ a standard ODE solving software and get the graph in Figure 6.

If we plot the expression

$$err(t) = y'(t) - (t^2 + 1)y(5/t) - 0.5y(t),$$

which is the difference between the two sides of Equation (36), we get a check of the solution. Figure 7 shows that the graphical solution shown in Figure 6 satisfies Equation(36) with an error of less than .002 over the open interval (1,5). An exact solution of Equation (36), of course, would show that the graph of $y = err(t)$ is simply the $t - axis$, asserting that $err(t) \equiv 0$.

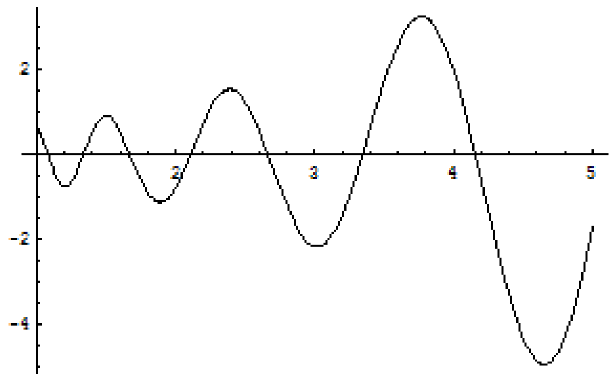


Figure 6: Solution to Equation (36)

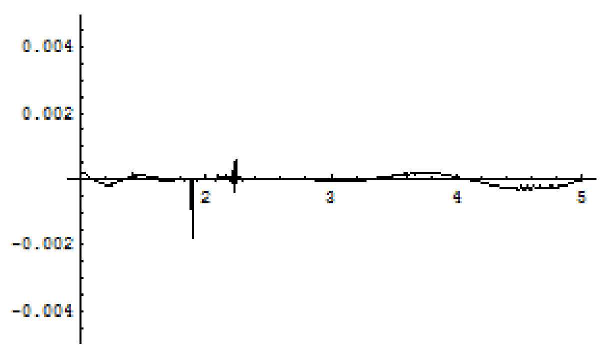


Figure 7: Graph of $y = err(t)$

Most of the problems that arise in applied fields cannot be solved exactly; therefore, numerical and graphical methods are widely used. Numerical solutions are approximations and computer algorithms necessarily produce solutions that contain errors, as illustrated in the graph in Figure 7.

The applied problems that can be modeled and solved by FDEs are varied and interesting; they range from traffic control (Bender), to robotics (Insperger), to chattering, or "tool vibrations when cutting steel " (Asi), to determining conditions prior to initial conditions (Nesbit), to climate models (Wu), and losses in transmission lines (Hale).

The theorem was proved for Equation (28) which is linear, but we sometimes see nonlinear IDEs used in various applications. For Example Linchuk[12], working in the "theory of automatic regulations" discussed the following nonlinear IDE

$$y'(t) + ay(t)y(-t) + b = C(\sin(ct) - \cos(ct)) \tag{38}$$

which can easily be solved by the method we described here. Simpy used the fact that $-t$ is its own inverse, and 0 is a fixed point. Then differentiate (38) getting equations that can be used to eliminate the $y'(-t)$ and the $y(-t)$ terms, much as was done in the above proof of the theorem for the linear case. The nonlinear case is discussed in my paper, Falbo[7].

In the paper by Ryder[16], a special case of Equation (28),

$$\begin{aligned} y'(t) &= \frac{1}{2} y(\frac{-t}{t+1}) \\ y(0) &= 1 \end{aligned} \tag{39}$$

is solved as $y(t) = \sqrt{t+1}$ for $t > -1$, with the fixed point being $t_0 = 0$. But the idempotent function, $-t/(t+1)$ also has a fixed point at $t_0 = -2$. With the initial condition $y(-2) = 1$ we would get, in addition to Ryder's solution, a second solution, $y(t) = \sqrt{-t-1}(1 - \ln(-t-1))$ for $t\epsilon(-\infty, -1)$.

In the next section we will be working with equations like (28), but for a more general class – one in which the deviating arguments are not idempotent functions, but rather decreasing functions with a fixed point. In this more general class we encounter a discontinuity in the derivative of the idempotent function, and this will, unfortunately, provide the computer with yet another opportunity to introduce errors and make the approximations somewhat less accurate. The good news, however, is that we will be able to see what kind of error we are actually dealing with. Also the solutions will be useful because they add to the list of FDEs that we can actually solve, albeit approximately and with restrictions on the domains of the solutions. It is by this process that we will make in-roads to what would otherwise would be an unexplored wilderness of hopeless problems.

# 4 Decreasing Functions at Their Fixed Points

We are now going to discuss another type of functional differential equations that can be solved by differentiation. As before, we confine our discussion to a real function of a real variable. The plan is to consider a function $f$, that is decreasing in a neighborhood of a fixed point, $t_0$, and the inverse, $f^{-1}$ which also has $t_0$ as a fixed point and is also decreasing in some neighborhood of $t_0$. It is possible to construct a new function $u(t)$ on some neighborhood of $t_0$ by making $u(t) = f(t)$ for all $t \leq t_0$ and $f^{-1}(t)$ if $t > t_0$. This new function $u(t)$ will be idempotent and we can show that the system

$$y'(t) = a(t)y(f(t)) + b(t)y(t), t < t_0 \quad (40\text{ a})$$
$$y'(t) = a(t)y(f^{-1}(t)) + b(t)y(t), t > t_0 \quad (40\text{ b})$$
$$y(t_0) = y_0$$

is equivalent to

$$y'(t) = a(t)y(u(t)) + b(t)y(t), t \neq t_0 \quad (41)$$
$$y(t_0) = y_0$$

as long as $t$ is in an interval that is a suitable domain for both $f$ and $f^{-1}$.

*EXAMPLE 6*
Let

$$f(t) = 1 - t^2, \text{ for } 0 < t < 1$$

The positive inverse of $f$ is

$$f^{-1}(t) = \sqrt{1-t}, \text{ for } 0 < t < 1$$

Each of these is a decreasing function on the open interval $(0, 1)$. They both have the same fixed point $t_0 = (-1 + \sqrt{5})/2$. We are excluding the endpoint 0, because $f'(0) = 0$, and we want $f'$ to be negative throughout its domain. Also since $f^{-1\prime}(t) = -1/(2\sqrt{1-t})$, which is not defined at $t = 1$, then we exclude 1 from the domain. Let us now define the following idempotent function

$$u(t) = f(t) \text{ if } 0 < t \leq t_0, \text{ and } u(t) = f^{-1}(t) \text{ if } t_0 \leq t < 1. \quad (42)$$

The function $u$ is continuous on $(0, 1)$, but is not differentiable at $t = t_0$ and in order to be able to make use of Equations (30) and (31), which require $u'(t)$ on the whole interval, we define $u'(t_0)$ as the average of $f'(t_0)$ and $f^{-1\prime}(t_0)$. Our solutions will be valid only on $(0, t_0) \cup (t_0, 1)$.

Another idempotent function, call it $v(t)$, may be defined by interchanging the functions $f$ and $f^{-1}$ thus,

$$v(t) = f^{-1}(t) \text{ if } 0 < t \leq t_0, \text{ and } v(t) = f(t) \text{ if } t_0 \leq t < 1 \quad (43)$$

with similar restrictions placed on any solution obtained using $v(t)$.

In Figures 8 and 9 we show the graphs of $y = u(t)$, and $y = v(t)$, respectively.

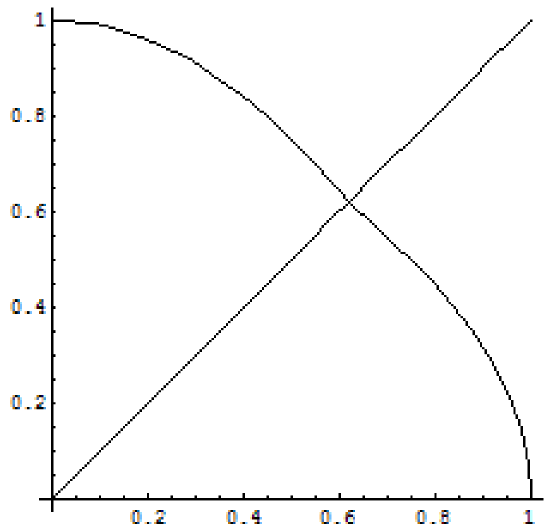


Figure 8: $y = u(t)$ and $y = t$

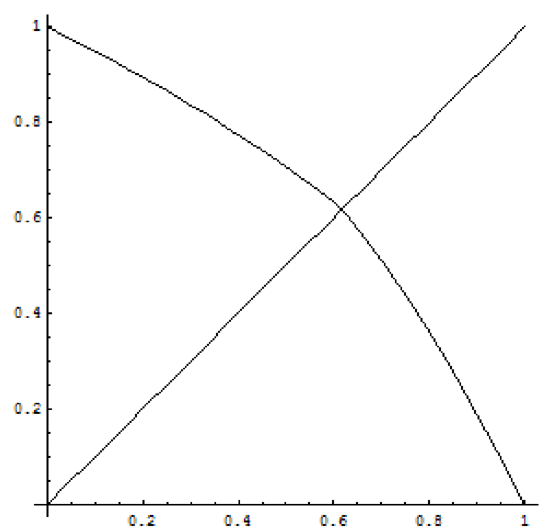


Figure 9: $y = v(t)$ and $y = t$

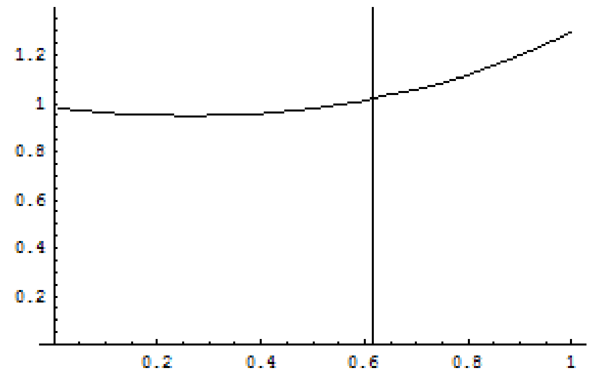


Figure 10: Solution to Equation (45)

Specifically, in this example, using the given $f$ and $f^{\,-1}$, and the function $u$ defined in (42), let us take $a(t) = -0.2$, $b(t) = t$, and $y_0 = 1$, then we want to solve.

$$y'(t) = -0.2y(1-t^2) + ty(t),\ t < t_0 \tag{44 a}$$

$$y'(t) = -0.2y(\sqrt{1-t}) + ty(t),\ t > t_0 \tag{44 b}$$

$$y(t_0) = 1$$

by solving

$$y'(t) = -0.2y(u(t)) + ty(t) \tag{45}$$

$$y(t_0) = 1$$

The troublesome endpoints 0 and 1 must be avoided in our numerical calculations; so, we solve the equation on a smaller domain. Let us use the closed interval $[0.01, 0.9999]$ rather than the open interval $(0, 1)$.

Applying Equations (30), (31), and (29) to Equation (45), we get the graphical solution shown in Figure 10

In Figure 10 the vertical line is at $t = t_0$.

Similarly, if we wish to solve Equations (44a) and (44b) with the functions $1 - t^2$ and $\sqrt{1-t}$ reversed, we would replace $u(t)$ by $v(t)$ in Equation (45).

**Conclusion** In this paper we have introduced two methods, MOC and STEPS for solving DDEs. Both of these methods can be taught in elementary courses in differential equations. We also observed that Remote Control Dynamical Systems can be treated as DDEs and solved accordingly. We have shown that dfferentiation can be used as a method for solving IDEs. In the last section we showed how to solve other types of FDEs, such as those whose deviating arguments are decreasing at their fixed points, as if they were IDEs.

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